

Quantum Resource Control for noisy EPR-steering with qubit measurements

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We demonstrate how quantum optimal control can be used to enhance quantum resources for bipartite one-way protocols, specifically EPR-steering with qubit measurements. Steering is relevant for one-sided device-independent key distribution, the realistic implementations of which necessitate the study of noisy scenarios. So far mainly the case of imperfect detection efficiency has been considered; here we look at the effect of dynamical noise responsible for decoherence and dissipation. In order to set up the optimisation, we map the steering problem into the equivalent joint measurability problem, and employ quantum resource-theoretic robustness monotones from that context. The advantage is that incompatibility (hence steerability) with arbitrary pairs of noisy qubit measurements has been completely characterised through an analytical expression, which can be turned into a computable cost function with exact gradient. Furthermore, dynamical loss of incompatibility has recently been illustrated using these monotones. We demonstrate resource control numerically using a special gradient-based software, showing, in particular, the advantage over naive control with cost function chosen as a fidelity in relation to a specific target. We subsequently illustrate the complexity of the control landscapes with a simplified two-variable scheme. The results contribute to the theoretical understanding of the limitations in realistic implementations of quantum information protocols, also paving way to practical use of the rather abstract quantum resource theories.

PACS numbers: 03.65.Ud, 02.60.Pn, 03.67.Mn, 03.67.Pp

Introduction— Due to the emerging technological motivation, it has become popular to view quantum effects as *resources* for tasks which cannot be described using classical physics [1–3]. While most work focuses on non-classical properties of quantum states, measurement resources are just as important, since the set of available measurements in any real experiment is restricted by the implementable *controls* such as laser pulses [4]. This is particularly relevant in correlation experiments where local parties make measurements on a shared entangled state. If the correlations violate a Bell inequality, they can be used in Quantum Key Distribution without any knowledge of the measurement devices; this is however experimentally difficult due to the detection loophole [6, 7]. Implementation is less challenging in a semi-device-independent scenario based on Einstein-Podolsky-Rosen (EPR) steering [5] (Fig. 1), which has attracted considerable interest recently [8–16]. It is intriguing as it requires entanglement but can be done with correlations admitting a hidden variable model. For instance, steering is possible with Gaussian states and measurements [5] which cannot violate Bell inequalities due to the hidden variable model provided by the Wigner function.

The intuitive idea is Alice “steering” Bob with her measurements through the shared state, which “transmits” an “assemblage” of conditional states to Bob [5, 8]. The maximally entangled state provides perfect transmission, and the general case reduces to that by replacing Alice’s measurements by certain state-dependent ones Bob reconstructs from the assemblage [16]. Hence, the quantum resource for steering can be described entirely by measurements; this is very useful when treating the loss of steerability due to local noise, as we see below. Interestingly, the required measurement resource has an

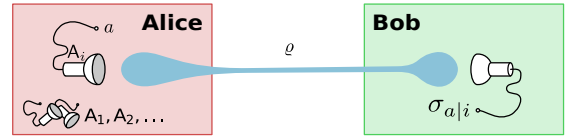


FIG. 1: A correlation experiment for EPR-steering.

independent meaning [14–16]: the measurements need to be *incompatible*. Here incompatibility does not mean non-commutativity, as noisy measurements are typically not projective. It means the non-existence of a “hidden” measurement jointly simulating all Alice’s measurements; this notion has been studied for a long time [17–25].

Hence, the loss of steerability can be described independently of the bipartite scenario, as the loss of incompatibility on Bob’s side. This leads to a simplification in system size, the simplest case being a single qubit. In order to *quantify* this loss, we need a numerical *incompatibility monotone*; they can be constructed [16, 26] using the noise-robustness idea from general quantum resource theories [1, 27–31]. The dynamics of incompatibility has recently been studied using these monotones [32].

In this paper, we take a new direction by showing how steering resource can be *directly* enhanced in the presence of Markovian noise, using numerical gradient-based quantum optimal control [4, 33, 34]. Research in this area aims at characterising operations reachable with restricted set of controls such as laser pulses, and numerically finding optimal pulses implementing a given target. While unitary control is fairly well established, control of noisy operations (quantum channels) is more challenging due to their complicated structure even in small sys-

tems [35–38]. In contrast to the usual optimisation of a distance from a specific target, we optimize over an incompatibility monotone, so as to do *purpose-oriented control* of EPR-steering, in analogy to entanglement control [39–41]. Steering is more challenging in small systems, where the existence of a suitable hidden variable model is already a nontrivial question; we use the special characterisation of qubit incompatibility [24, 25] to compute a faithful incompatibility monotone with exact gradient. The purpose-oriented control of steering rather than targeting specific measurements is motivated by the fact that *many* measurements have equal steering potential, and that targeting specific ideal (projective) ones is not likely to work as they are usually not reachable in noisy systems. The problem is also intriguing in that the monotones are unitary-invariant; unitary control *can only help in the presence of noise* which destroys the resource in the first place; hence it is a priori not at all clear if it actually does help.

The quantum resource for steering— We look at the bipartite scenario with Alice and Bob sharing a state ρ on the tensor product Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. Alice has a restricted set $\mathcal{M}_A = \{A_1, \dots, A_n\}$ of measurements, assumed to be general (possibly non-projective) POVMs, as this is necessary in noisy scenarios. Hence each A_i stands for a collection of positive-semidefinite matrices $A_i(a)$ with $\sum_a A_i(a) = \mathbb{1}$. The steering protocol proceeds as follows [5]: Alice chooses index i , performs A_i on her system getting an outcome a , and announces (i, a) to Bob, who uses state tomography to extract the “assemblage” [9] $\sigma_{a|i} := \text{tr}_A[\rho(A_i(a) \otimes \mathbb{1})]$ of conditional states. The assemblage is called steerable if Bob cannot reconstruct it from some pre-existing collection of “hidden” states using only classical information on Alice’s results. We state the precise definition in terms of the associated joint measurability problem [14–16]: following [16] we restrict w.l.o.g. $\mathcal{H}_B = \text{ran} \rho_B$, and define

$$S_\rho(A) := \rho_B^{-\frac{1}{2}} \text{tr}_A[\rho(A \otimes \mathbb{1})] \rho_B^{-\frac{1}{2}},$$

so that S_ρ is a unital CP map between Alice’s and Bob’s observable algebras, and $\text{tr}[\rho(\mathbb{1} \otimes S_\rho(A^\top))] = \text{tr}[\rho(A \otimes \mathbb{1})]$ for any matrix A , where A^\top is the transpose. This means that Bob can simulate Alice’s measurements via the POVMs $B_i^\rho = S_\rho(A_i^\top)$ determined by the assemblage. The setting is *non-steerable* if these are *jointly measurable* in that each B_i^ρ is a classical probabilistic postprocessing of a single POVM $G(\lambda)$; formally,

$$B_i^\rho(a) = \sum_{\lambda} p(a|i, \lambda) G(\lambda), \quad \text{for all } i = 1, \dots, n,$$

where $\sum_a p(a|i, \lambda) = 1$. This definition is equivalent to the usual notion of joint measurability via marginals [13].

In conclusion, the quantum resource for EPR-steering can be characterised as the opposite of joint measurability,

often called *incompatibility*, of the collection

$$B_i^\rho = S_\rho(A_i), \quad i = 1, \dots, n$$

of measurements. This formulation has the advantage of referring only to a single system; the “nonlocal” aspect is encapsulated in S_ρ . This is especially useful in describing local noise: given a channel Λ on Alice’s side changing the state as $\rho \mapsto \tilde{\rho} = (\Lambda \otimes \text{Id})(\rho)$, the assemblage changes

$$\sigma_{a|i} \mapsto \tilde{\sigma}_{a|i} = \text{tr}_A[\rho(\Lambda^*(A_i(a)) \otimes \mathbb{1})].$$

Then the resource transforms into

$$\tilde{B}_i^\rho = S_\rho \circ \Lambda^*(A_i), \quad i = 1, \dots, n, \quad (1)$$

the Heisenberg channel Λ^* simply concatenating with S_ρ . This has a clear interpretation: S_ρ describes *preparation noise* in an imperfect production of a maximally entangled state (for which $S_\rho = \text{Id}$), while Λ is the subsequent *dynamical noise*. Steerability of the noisy assemblage is equivalent to the incompatibility of (1).

Resource theories also contain the idea of *quantification* [1]. Incompatibility of a collection (B_1, \dots, B_n) of measurements can be quantified by an *incompatibility monotone* [26], i.e. a number $\mathcal{I}(B_1, \dots, B_n)$ which is zero exactly in the jointly measurable case, and

$$\mathcal{I}(\Lambda^*(B_1), \dots, \Lambda^*(B_n)) \leq \mathcal{I}(B_1, \dots, B_n) \quad (2)$$

for any positive linear map Λ . This fits well with the steering resource (1), where the total channel $S_\rho \circ \Lambda^*$ then effects a *quantitative* loss of the resource. In particular, loss due to continuous dynamics $t \mapsto \Lambda_t$ is described by the function $t \mapsto \mathcal{I}(S_\rho \circ \Lambda_t^*(A_1), \dots, S_\rho \circ \Lambda_t^*(A_n))$.

Good operational incompatibility monotones describe convex-geometric *noise-robustness* [1, 16, 26, 31]: we mix *classical* noise with distribution $p = (p_i)$ into measurements via $A \mapsto (1 - \lambda)A + \lambda p \mathbb{1}$, and define \mathcal{I} to be the minimal λ for which the resource is lost, i.e. measurements become jointly measurable (see [26, 32] for discussion).

In this paper, we look at the simplest setting with $\mathcal{H}_A = \mathbb{C}^2$ and two measurements for Alice; this case is already interesting. The speciality is that robustness monotones can be computed using the analytical characterisation of incompatibility [24, 25]: Given a POVM $A = (A, \mathbb{1} - A)$ on \mathbb{C}^2 , we identify $A = \frac{1}{2}(x^0 \mathbb{1} + \mathbf{x} \cdot \boldsymbol{\sigma})$ with the 4-vector $x = (x^0, \mathbf{x})$, and $\mathbb{1} - A$ with $x^\perp := (2 - x^0, -\mathbf{x})$. The condition $0 \leq A \leq \mathbb{1}$ reads $x, x^\perp \in \mathcal{F}_+$, where $\mathcal{F}_+ = \{x \mid \langle x|x \rangle \geq 0, x^0 \geq 0\}$ and $\langle x|y \rangle := x^0 y^0 - \sum_{i=1}^3 x^i y^i$ is the Minkowski form. A pair of measurements x_1 and x_2 is jointly measurable *if and only if* $\mathcal{C}(x, y) \geq 0$, where

$$\begin{aligned} \mathcal{C}(x_1, x_2) := & [\langle x_1|x_1 \rangle \langle x_1^\perp|x_1^\perp \rangle \langle x_2|x_2 \rangle \langle x_2^\perp|x_2^\perp \rangle]^{1/2} \\ & - \langle x_1|x_1^\perp \rangle \langle x_2|x_2^\perp \rangle + \langle x_1|x_2^\perp \rangle \langle x_1^\perp|x_2 \rangle + \langle x_1|x_2 \rangle \langle x_1^\perp|x_2^\perp \rangle. \end{aligned}$$

We use the robustness monotone $\mathcal{I}_b(x, y)$ of [26]: given a probability $p = \frac{1}{2}(1 + b)$, the above classical noise is

$x \mapsto N_{\lambda,b}(x) := ((1 - \lambda)x^0 + \lambda 2p, (1 - \lambda)\mathbf{x})$, and $\mathcal{I}_b(x, y)$ is by definition the unique solution $0 \leq \lambda \leq 1/2$ of

$$\mathcal{C}(N_{\lambda,b}(x_1), N_{\lambda,b}(x_2)) = 0. \quad (3)$$

Interestingly, $\mathcal{I}_0(x_1, x_2)$ coincides with the maximal violation of the CHSH-Bell inequality with Alice's measurements (x_1, x_2) [26]. This monotone was recently used in studying the loss of incompatibility on open systems [32].

Optimal resource control of steering— Having identified and quantified the resource for steering and described its loss in open systems, it is natural to ask if this loss can be slowed down by control. As discussed in the introduction, our approach is to directly optimise the incompatibility resource (1) using the monotones \mathcal{I}_b .

We take Alice's noise to be Markovian: $\Lambda_t = e^{t\mathcal{L}_0}$ with a drift Lindbladian \mathcal{L}_0 . We look at two basic cases, amplitude damping $\mathcal{L}_0^{\text{AD}}(\rho) = \gamma(2\sigma_- \rho \sigma_+ - \{\sigma_+ \sigma_-, \rho\})$ (containing decoherence and dissipation), and dephasing in the σ_y -basis $\mathcal{L}_0^{\text{DP}}(\rho) = \gamma(\sigma_y \rho \sigma_y^\dagger - \rho)$ (only decoherence), with $\gamma = 0.1$. We model control (e.g. a laser pulse) by changing \mathcal{L}_0 into

$$\mathcal{L}_c = \mathcal{L}_0 - ic[H, \cdot],$$

where the H is a control Hamiltonian and $c \in \mathbb{R}$. Applying a sequence $\mathbf{c} = (c_1, \dots, c_m)$ of such pulses, each of duration Δt , the dynamics at time $T = m\Delta t$ is $\Lambda_{\mathbf{c}} = e^{\Delta t \mathcal{L}_{c_m}} \dots e^{\Delta t \mathcal{L}_{c_1}}$. We consider the setting where Alice aims at steering Bob at time T with a measurement pair (x_1, x_2) , given that the initial state was ρ . According to the preceding section, the information on the resource is faithfully encoded into the cost function

$$f(\mathbf{c}) = \mathcal{I}_b(S_\rho \circ \Lambda_{\mathbf{c}}^*(x_1), S_\rho \circ \Lambda_{\mathbf{c}}^*(x_2)) \quad (4)$$

describing the steering robustness. In particular, this function is nonzero if and only if the setting is steerable. We implemented (4) numerically by solving (3) using standard root-finding which needs only a few iterations as the function is just a combination of polynomials and square root, and changes sign on $[0, 1/2]$. Given this value, the gradient of \mathcal{I}_b can be found *analytically* via implicit differentiation. This fits particularly well with the control software *Qtrl* [42], which implements the Frechet derivative of $\mathbf{c} \mapsto \Lambda_{\mathbf{c}}$ to be used in computing cost functions; hence we get the gradient of $f(\mathbf{c})$ from the chain rule, and employ optimisation based on exact gradient.

We take a maximally entangled ρ so that $S_\rho = \text{Id}$, Alice's measurements (σ_x, σ_z) , $H = \sigma_y + \sigma_z$, and use the monotone \mathcal{I}_0 . The results for different times T are depicted in Fig. 2. It shows the robustness \mathcal{I}_0 in the uncontrolled case, the optimised value, and comparison with the naive control strategy aiming at the channel closest to the identity. The computations were done with *Qtrl* on HPC Wales with $m = 20$ (number of time slots) and optimised over 100 random initial pulses. Optimisation

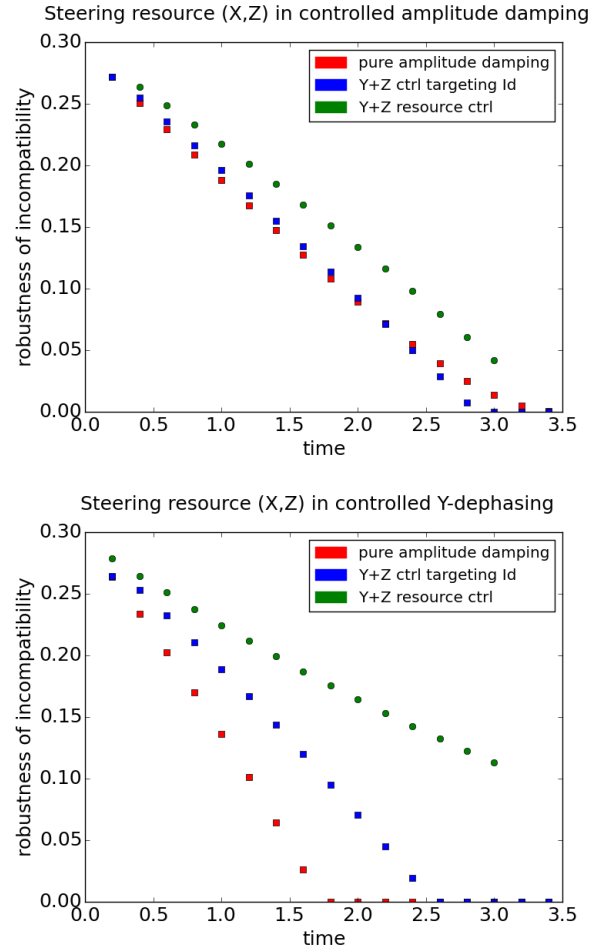


FIG. 2: Resource control of steering under amplitude damping (upper panel) and dephasing in the σ_y -basis (lower panel).

with the cost function (4) is considerably slower than the naive method, however the results are significantly better. We also see that control works better with dephasing, presumably due to the lack of dissipation present in the amplitude damping.

An inspection of the optimal pulses showed that the amplitudes peak close to the end, suggesting that only a few time slots are needed. Accordingly, we considered the following simple scheme: drift until time $t_{\text{drift}} < T$, then apply two pulses c_1, c_2 , each of duration $\Delta t = (T - t_{\text{drift}})/2$. This implements the map $\Lambda_{c_1, c_2} = e^{\Delta t \mathcal{L}_{c_2}} e^{\Delta t \mathcal{L}_{c_1}} e^{t_{\text{drift}} \mathcal{L}_0}$. The corresponding control landscape for $f(c_1, c_2)$ in (4) is shown in Fig. 3 for the amplitude damping case with $t_{\text{drift}} = 2.6$ and $T = 2.8$. At this time resource control can boost steering robustness up to 0.062, which is more than a factor of two improvement from the pure noise (see Fig. 2). The maximum is attained with the pulse $\mathbf{c} = (-1.42, 12.32)$. Clearly the choice of controls is crucial; looking also at Figure 2 we observe that both pure noise (no control) and the naive

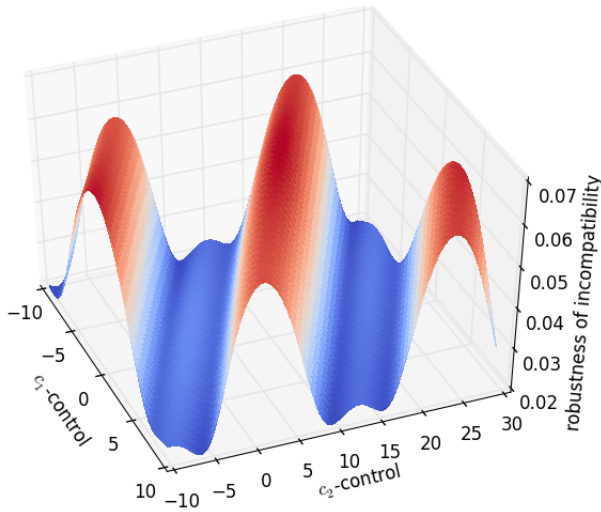


FIG. 3: A control landscape of steering robustness under amplitude damping noise. The maximum represents optimal quantum resource control.

strategy are far from the maximum. In the dephasing case the landscape is similar, except that there are controls leading to non-steerability (including the no control and naive control cases), while optimal control manages to preserve steering. The maximum steering robustness is 0.125 with the pulse $\mathbf{c} = (1.80, -12.88)$. It appears to be crucial that the controls are applied just before measuring, when incompatibility is close to its maximum.

Conclusion and outlook— We have demonstrated how steering can be enhanced by control in noisy qubit systems, with direct resource optimisation of (4) performing better than target-based one. The effect of an imperfect initial state ρ , and other non-Markovian features [32] remain to be studied. Our results pave the way for general schemes for implementing optimal noisy quantum resources in controlled open systems. The optimisation naturally becomes slow in large systems, with analytical gradient no longer available. Nevertheless, (4) can always be computed efficiently via a semidefinite program [26], and approximations based on steering inequalities [10] could be used in analogy to the entanglement control [40] to make the computations feasible.

Acknowledgment— We thank the QuTiP project [42] for providing the software, (especially A. Pitchford for developing the pulse optimisation package), and HPC Wales for computing time. We acknowledge financial support from the EPSRC project EP/M01634X/1.

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